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COMMENT

How to define systematically all possible two-particle state vectors in terms of conditional probabilities

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Abstract. All possible two-particle state vectors $|\phi\rangle_{12}$ can be generated in a systematic way from first-particle-reduced statistical operators ρ_1 and anti-unitary correlation operators. It is pointed out that this procedure is easily completed so that $|\phi\rangle_{12}$ is defined in terms of ρ_1 and conditional statistical operators for the second particle.

To explain the problem addressed in this comment, we turn to classical discrete probability theory. Let M and N be two sets of at most countably infinite cardinality, and p_{mn} probability distributions on $M \times N$ ($\forall m, \forall n$: $p_{mn} > 0$; $\Sigma_{mn} p_{mn} = 1$). As is well known, $p_m = \Sigma_n p_{mn}$ is the marginal (distribution) on M, and for every $m \in M$, $p_m > 0$: $p(n|m) \equiv p_{mn}/p_m$ is the corresponding conditional probability on N. Now, if we want to define systematically all possible p_{mn}, a natural way to do this is given by the following construction.

(a) The first step is to choose an arbitrary marginal p_m, and for different choices the final p_{mn} are certainly different.

(b) In the second step (we actually have a set of substeps), for each $m \in M$, $p_m > 0$, we choose an arbitrary probability distribution $p(n|m)$ for the conditional probability, and for distinct choices we obtain distinct $p_{mn} \equiv p_m p(n|m)$.

Observe the three important features of this construction.

(i) The first step is the definition of the marginal.

(ii) The nth step may depend on the $(n-1)$ preceding ones, but not on the succeeding ones.

(iii) A different choice at each step (or substep) necessarily gives different final p_m.

The problem is how to follow this procedure of construction in quantum mechanics. It is the purpose of this comment to point out that along the lines (i)-(iii) one can define all two-particle (or two-subsystem) state vectors $|\phi\rangle_{12} \in \mathscr{H}_1 \otimes \mathscr{H}_2$ ($\mathscr{H}_i$ being separable Hilbert spaces, $i = 1, 2$). I believe this is of some importance for a systematic study of quantum mechanical correlations.

It was shown in previous work (see corollary 1 in subsection 2.1 of Herbut 1986) that any statistical operator ρ_{12} in $\mathscr{H}_1 \otimes \mathscr{H}_2$ ($\rho_{12} \geqslant 0$, $\mathrm{Tr}_{12}\rho_{12} = 1$) is defined in terms of its reduced statistical operator in $\mathscr{H}_1$

$$\rho_1 \equiv \mathrm{Tr}_2\, \rho_{12} \tag{1}$$

(Tr_2 being the partial trace in $\mathscr{H}_2$; ρ_1 is the counterpart of the p_m above), and the set of all conditional statistical operators (the counterparts of the $p(n|m)$) to which ρ_{12} gives rise:

$$\forall P_1 \in \mathscr{P}(\mathscr{H}_1),\ p \equiv \mathrm{Tr}_1\, P_1\rho_1 > 0 \qquad \rho_2(P_1) \equiv p^{-1}\, \mathrm{Tr}_2\, (P_1 \otimes 1)\rho_{12} \tag{2}$$

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where $\mathscr{P}(\mathscr{H}_1)$ is the set of all projectors in $\mathscr{H}_1$. Actually, it was shown that if $\rho_{12} \neq \rho'_{12}$, then either $\rho_1 \neq \rho'_1$ (cf (1)), and/or

$$\exists P_1 \in \mathscr{P}(\mathscr{H}_1) \qquad \mathrm{Tr}_1\, \rho_1 P_1 > 0 \qquad \rho_2(P_1) \neq \rho'_2(P_1)$$

(cf (2)).

The physical meaning of $\rho_2(P_1)$ is that it represents the state of the second particle under the condition that the quantum event P_1 occurred (its characteristic value 1 was obtained in first-kind or second-kind measurement) on the first particle.

In the case of general statistical operators ρ_{12} it is far from clear how one could give a construction procedure along the lines (i)–(iii). But it seems a natural assumption that it should be feasible.

As far as pure-state statistical operators $\rho_{12} \equiv |\phi\rangle_{12}\langle\phi|_{12}$ are concerned, there is another relevant previous result (see theorems 4, 5 and 7 in Herbut and Vujičić (1976)). Each state vector $|\phi\rangle_{12} \in \mathscr{H}_1 \otimes \mathscr{H}_2$ can be constructed along the lines (i)–(iii) (but not in terms of conditional statistical operators).

(*a*) One chooses an arbitrary statistical operator ρ_1 in $\mathscr{H}_1$ to be the first-particle-reduced statistical operator. Different choices lead eventually to different $|\phi\rangle_{12}$.

(*b*) One chooses an antiunitary map of $R(\rho_1)$ (the range of ρ_1) into $\mathscr{H}_2$: U_a (the so-called correlation operator of $|\phi\rangle_{12}$). Distinct choices result in distinct $|\phi\rangle_{12}$.

Finally, one may take an eigenbasis $\{|i\rangle_1 : \forall i\}$ of ρ_1 spanning $R(\rho_1)$, and then

$$|\phi\rangle_{12} = \Sigma_i r_i^{1/2} |i\rangle_1 \otimes (U_a |i\rangle_1) \tag{3}$$

(all such sub-bases give one and the same $|\phi\rangle_{12}$). Here r_1 is the chacteristic value of ρ_1 corresponding to $|i\rangle_1$, $i = 1, 2, \ldots$.

All we do now is introduce the conditional statistical operators. Let $|\varphi\rangle_1 \in \mathscr{H}_1$ be outside the null space of ρ_1 (otherwise $\mathrm{Tr}_1 \rho_1 |\varphi\rangle_1\langle\varphi| = 0$, and $\mathrm{Tr}_2(|\varphi\rangle_1\langle\varphi|_1 \otimes 1)|\phi\rangle_{12}\langle\phi|_{12} = 0$ as seen from (3)). We define

$$|\chi\rangle_2 \equiv U_a \rho_1^{1/2} |\varphi\rangle_1 / \| U_a \rho_1^{1/2} |\varphi\rangle_1 \|. \tag{4}$$

Then

$$\rho_2(|\varphi\rangle_1\langle\varphi|_1) = |\chi\rangle_2\langle\chi|_2$$

where the left-hand side is the conditional statistical operator from $|\phi\rangle_{12}$ that is given by (3). (For the proof see equation (34) and theorem 1 in Herbut and Vujičić 1976.)

For an aribtrary $P_1 \in \mathscr{P}(\mathscr{H}_1)$, we first decompose it into orthogonal ray projectors:

$$P_1 = \Sigma_k |\varphi_k\rangle\langle\varphi_k|. \tag{5}$$

Since

$$\begin{aligned} \rho_2(P_1) &\equiv p^{-1} \mathrm{Tr}_2(1 \otimes \Sigma_k |\varphi_k\rangle\langle\varphi_k|)\rho_{12} \\ &= \Sigma'_k (\mathrm{Tr}_1\, \rho_1 |\varphi_k\rangle\langle\varphi_k| / \mathrm{Tr}_1 \rho_1 P_1) \rho_2(|\varphi_k\rangle\langle\varphi_k|) \end{aligned}$$

(the prime denotes that the $|\varphi_k\rangle$ from the null space of ρ_1 are omitted), we finally have

$$\rho_2(P_1) = \Sigma'_k (\mathrm{Tr}_1\, \rho_1 |\varphi_k\rangle\langle\varphi_k| / \mathrm{Tr}_1\, \rho_1 P_1) |\chi_k\rangle\langle\chi_k| \tag{6}$$

where $|\chi_k\rangle$ is determined by $|\varphi_k\rangle$ via (4). The operator $\rho_2(P_1)$ does not depend on the choice of the decomposition (5).

Thus ρ_1 and the correlation operator U_a define the entire requisite set of conditional statistical operators $\{\rho_2(P_1) : \forall P_1 \in \mathscr{P}(\mathscr{H}_1), \mathrm{Tr}_1\, \rho_1 P_1 > 0\}$. But to define all $\rho_{12} \equiv |\phi\rangle_{12}\langle\phi|_{12}$ systematically, one can hardly do better than go via (3) as explained above.

This construction of $|\phi\rangle_{12}$ in terms of the correlation operator U_a may come as a surprise to the reader unfamiliar with the antilinear-operator approach to two-particle state vector theory, in particular to the theory of distant correlations (Herbut and Vujičić 1976, 1987, Vujičić and Herbut 1984, 1988). To lessen the surprise, we make a final comment. Returning to classical discrete probability theory, there is a class of p_{mn} analogous to the $|\phi\rangle_{12}$. Their construction goes as follows.

(a) We choose an arbitrary probability distribution p_m on M to be the marginal.

(b) We choose an arbitrary bijection U of the so-called support $\bar{M} \equiv \{m: p_m > 0\}$ of p_m onto some subset of N. Then we define

$$\forall m, p_m > 0 \qquad p(n|m) \equiv \delta_{n,U(m)}.$$

In information theory one calls these

$$p_{mn} \equiv p_m \delta_{n,U(m)}$$

noiseless and lossless channels.

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